Thus the only operation special to the path is the calculation of the right-hand side of (12).

Now y is determined as a function of ψ by the preceding graphical method. Accordingly within each range (V_p to V_q) or portion of such range,

$$\int f(y)(\sec\psi)^{n_{pq}+1}d\psi$$

can be calculated by any convenient method of quadrature. Thus u (and v) are determined as functions of ψ, and all the elements of the path (viz., (x, y and t) are determined in terms of ψ.

The Lommel-Weber Ω Function and its Application to the Problem of Electric Waves on a Thin Anchor Ring.

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The function $\Omega_n(x)$ defined by the integral $\dfrac{1}{\pi}\displaystyle\int_0^{\pi}\sin(x\sin\phi - n\phi)\,d\phi$, or, when n is an even integer, by $\dfrac{2}{\pi}\displaystyle\int_0^{\pi/2}\sin(x\sin\phi)\cos n\phi\,d\phi$, is closely related to $J_n(x)$, which was first given by Bessel under the form $\dfrac{1}{\pi}\displaystyle\int_0^{\pi}\cos(x\sin\phi - n\phi)\,d\phi$.

The Ω function, although of less importance in its application than the J function, occurs in a number of problems in mathematical physics, more especially those relating to the interference and diffraction of light.

The function $\Omega_0(x) = \dfrac{2}{\pi}\displaystyle\int_0^{\pi/2}\sin(x\sin\phi)\,d\phi$ was employed by Lord Rayleigh* in the problem of the reaction of the air on a vibrating circular plate, and by H. Struve† in the theory of diffraction in telescopes. H. F. Weber‡ applied the function $\Omega_{\frac{1}{2}}(x)$ and, in addition, found a number of interesting properties of $\Omega_\nu(x)$, including the recurrence formula and the development in series of ascending and descending powers of the argument.

* Lord Rayleigh, 'Theory of Sound,' vol. 2, p. 164, equation (5).

† H. Struve, "Beitrag zur Theorie der Diffraction an Fernröhren," 'Annalen der Physik und Chemie,' vol. 17, p. 1013.

‡ H. F. Weber, "Die wahre Theorie der Fresnel'schen Interferenz-Erscheinungen," 'Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich,' vol. 24, p. 48.

Lommel* proved that the integral $\frac{1}{\pi}\int_0^\pi \sin(\nu\phi - x\sin\phi)\,d\phi$ is represented by the following series when ν is an even number, $2n$,

$$\Omega_{2n}(x) = -\frac{2}{\pi}s_{0,2n}(x) \qquad \text{or} \qquad -\frac{2}{\pi}[S_{0,2n}(x)+Y_{2n}(x)],$$

where

$$s_{0,\nu}(x) = \frac{x}{(1^2-\nu^2)} - \frac{x^3}{(1^2-\nu^2)(3^2-\nu^2)} + \frac{x^5}{(1^2-\nu^2)(3^2-\nu^2)(5^2-\nu^2)} - \ldots,$$

$$S_{0,\nu}(x) = \frac{1}{x} - \frac{(1^2-\nu^2)}{x^3} + \frac{(1^2-\nu^2)(3^2-\nu^2)}{x^5} - \ldots,$$

and $Y_{2n}(x)$ is the Bessel function of the second kind according to Lommel's definition. A similar formula gives the value of the function when ν is odd.

Functions of higher order occur in the problem of electric waves on a thin anchor ring. If a is the radius of the circular axis of the ring, ϵ the radius of the circular section, m any integer and $\Pi = e^{ia\rho}\, e^{ipt}/\rho$, Lord Rayleigh† found that approximately

$$\int_0^\pi \frac{(a^2\alpha^2\cos\phi - m^2)\cos m\phi\, d\phi}{\sqrt{(\epsilon^2+4a^2\sin^2\frac{1}{2}\phi)}} + \int_0^\pi \frac{(e^{2i\alpha\sin\frac{1}{2}\phi}-1)(a^2\alpha^2\cos\phi - m^2)\cos m\phi\, d\phi}{2a\sin\frac{1}{2}\phi} = 0. \quad (1)$$

The second integral determines the imaginary part of $(a^2\alpha^2 - m^2)L$, where $L = \log\frac{8a}{\epsilon}$, and it has been shown‡ that this can be expressed in terms of Bessel functions of nearly equal order and argument, viz.

$$\frac{im^2\pi}{2}\{J_{2m-1}(2m) - J_{2m+1}(2m)\} \qquad \text{or} \qquad im^2\pi \,.\, \frac{d}{dx}\,.\, J_{2m}(x), \quad (2)$$

and $x = 2m$.

On the other hand, the real part of $(a^2\alpha^2 - m^2)L$ is made up of contributions from both integrals. The second integral takes the form

$$\frac{m^2}{2a}\int_0^{\pi/2} d\psi \frac{e^{ix\sin\psi}-1}{\sin\psi}\{\cos(2m+2)\psi - 2\cos 2m\psi + \cos(2m-2)\psi\}, \quad (3)$$

where $x^2 = 4a^2\alpha^2 = 4m^2$ nearly. To evaluate this, Lord Rayleigh introduces the integral

$$\frac{2}{\pi}\int_0^{\pi/2} d\psi\cos 2m\psi \,.\, e^{ix\sin\psi} = J_{2m}(x) + iK_{2m}(x). \quad (4)$$

* Lommel, "Zur Theorie der Bessel'schen Functionen," 'Math. Ann.,' vol. 16, p. 188.

† Lord Rayleigh, "Electrical Vibrations on a Thin Anchor Ring," 'Roy. Soc. Proc.,' vol. 87, p. 194.

‡ *Loc. cit.*, p. 196.

The K function is no other than the Ω function, but the term -1 in the numerator $e^{ix\sin\psi}-1$ in (3), although not affecting the imaginary part involving the J functions, reduces the result for the real part to

$$m^2\pi\left\{\frac{d}{dx}\,.\,\Omega_{2m}(x)-(-1)^m\frac{2}{\pi(2m-1)(2m+1)}\right\}.$$

But the contribution to the real part of $(a^2\alpha^2-m^2)\,\mathrm{L}$ from the first integral is

$$(-1)^m\,.\,\frac{2m^2}{(2m-1)(2m+1)}=(-1)^m m^2\pi\,.\,\frac{d}{dx}\frac{x/2}{\Gamma(m+\frac{3}{2})\Gamma(-m+\frac{3}{2})}$$

and $(-1)^m\dfrac{x/2}{\Gamma(m+\frac{3}{2})\Gamma(-m+\frac{3}{2})}$ is the first term of $\Omega_{2m}(x)$.

Consequently, equation (21) in Lord Rayleigh's paper takes the simple form

$$(a^2\alpha^2-m^2)\,\mathrm{L}=m^2\pi\{i\mathrm{J}_{2m}'(x)+\Omega_{2m}'(x)\}\quad\text{and}\quad x=2m, \tag{5}$$

$$=\frac{m^2\pi}{2}\{i[\mathrm{J}_{2m-1}(2m)-\mathrm{J}_{2m+1}(2m)]+[\Omega_{2m-1}(2m)-\Omega_{2m+1}(2m)]\}, \tag{6}$$

since*

$$\Omega_{2m-1}(x)-\Omega_{2m+1}(x)=2\,.\,\frac{d}{dx}\,.\,\Omega_{2m}(x). \tag{7}$$

Hence

$$-\alpha=\frac{m}{a}\left\{1+\frac{\pi}{4\mathrm{L}}\{i[\mathrm{J}_{2m-1}(2m)-\mathrm{J}_{2m+1}(2m)]+[\Omega_{2m-1}(2m)-\Omega_{2m+1}(2m)]\}\right\} \tag{8}$$

and the equivalent wave lengths λ_m are given by

$$\lambda_m=\frac{2\pi a}{m}\left\{1-\frac{\pi}{4\mathrm{L}}[\Omega_{2m-1}(2m)-\Omega_{2m+1}(2m)]\right\}. \tag{9}$$

Where m is small, $\Omega_{2m-1}(2m)$ and $\Omega_{2m+1}(2m)$ are readily obtained from the series in ascending powers of m,

$$\Omega_{2m+1}(x)=(-1)^{m+1}\sum_{s=0}^{s=\infty}\frac{(-1)^s(x/2)^{2s}}{\Gamma(s+m+\frac{3}{2})\Gamma(s-m+\frac{1}{2})}, \tag{10}$$

or, in terms of the Bessel functions† $\mathrm{J}_{2s}(x)$,

$$\Omega_{2m+1}(x)=-\frac{2}{\pi}\left[\frac{\mathrm{J}_0(x)}{2m+1}+(4m+2)\sum_{s=1}^{s=\infty}\frac{\mathrm{J}_{2s}(x)}{(2m+1)^2-4s^2}\right]. \tag{11}$$

The asymptotic expansion given by Lommel and Weber‡ cannot be usefully employed where the order and argument are large. Expressions giving the values of $\Omega_n(n)$, $\Omega_{n-1}(n)$, etc., can be derived from a result due to Sonine and Schläfli.

* Nielsen, 'Cylinderfunktionen,' p. 49.

† Nielsen, 'Cylinderfunktionen,' p. 67.

‡ *Loc. cit.*, p. 48.

Sonine's result is*

$$Y_n(x) = \int_0^\pi \sin(x \sin\phi - n\phi)\, d\phi - \int_0^\infty e^{-x\sinh\theta}\left[e^{n\theta} + (-1)^n \cdot e^{-n\theta}\right] d\theta,$$

the function $Y_n(x)$ being the same as Hankel's and equal to $-2G_n(x)$.

Schläfli found that†

$$Y_n(x) - (\log 2 - \gamma)\, J_n(x) = \tfrac{1}{2}\int_0^\pi \sin(x\sin\phi - n\phi)\, d\phi$$

$$-\tfrac{1}{2}\int_0^\infty e^{-x\sinh\theta}\left[e^{n\theta} + (-1)^n e^{-n\theta}\right] d\theta.$$

Here $Y_n(x) - (\log 2 - \gamma)\, J_n(x)$ is equal to $-G_n(x)$.

Expressed in the form

$$\Omega_n(x) = Y_n(x) + \frac{1}{\pi}\int_0^\infty e^{-x\sinh\theta}\left(e^{n\theta} + \cos n\pi \cdot e^{-n\theta}\right) d\theta, \tag{12}$$

the Neumann function $Y_n(x)$ is equal to $-2/\pi \cdot G_n(x)$. Very complete Tables‡ of $G_n(n)$ and $G_{n-1}(n)$ with other related functions have been computed for a large number of values of n. The integral in (12) can be evaluated by the "Saddle Point" method of contour integration,§ or by the less elaborate method given below. Taking the case where the order and argument are equal and n is an even integer,

$$\Omega_n(n) = -\frac{2}{\pi} G_n(n) + \frac{1}{\pi}\int_0^\infty e^{-n\sinh\theta}\left(e^{n\theta} + e^{-n\theta}\right) d\theta. \tag{13}$$

In the integral $\dfrac{1}{\pi}\displaystyle\int_0^\infty e^{-n(\sinh\theta - \theta)}\, d\theta$, change the variable by substituting t for $\sinh\theta - \theta$; then if λ is written for $(6t)^{1/3}$,

$$\lambda = \theta + \frac{\theta^3}{60} + \frac{\theta^5}{8400} - \dots.$$

Reversing this series,

$$\theta = \lambda - \frac{\lambda^3}{60} + \frac{\lambda^5}{1400} - \frac{\lambda^7}{25200} + \dots$$

and

$$d\theta = \left(1 - \frac{\lambda^2}{20} + \frac{\lambda^4}{280} - \frac{\lambda^6}{3600} + \dots\right) d\lambda.$$

Writing α for $\left(\dfrac{6}{n}\right)^{1/3}$ and applying $\displaystyle\int_0^\infty e^{-nt}\, t^{p-1}\, dt = \frac{\Gamma(p)}{n^p}$ to each of the

* Sonine, "Fonctions Cylindriques," 'Math. Ann.,' vol. 3, p. 27.

† Schläfli, "Bessel'schen Functionen," 'Math. Ann.,' vol. 3, p. 143.

‡ 'Report of the Mathematical Tables Committee of the British Association,' 1916, pp. 92–96.

§ Debye, 'Math. Ann.,' vol. 67, pp. 535–558; Brillouin, 'Annales de l'École Normale,' vol. 33, pp. 17–69.

terms in the integrand, we find

$$\frac{1}{\pi}\int_0^\infty e^{-n(\sinh\theta-\theta)}\,d\theta$$

$$=\frac{1}{3\pi}\int_0^\infty e^{-nt}\left(6^{1/3}t^{-2/3}-\frac{6}{20}+\frac{1}{280}\,6^{5/3}\,t^{2/3}-\frac{1}{3600}\,6^{7/3}\,t^{4/3}+\dots\right)dt$$

$$=\frac{1}{3\pi}\left[\alpha\,.\,\Gamma(\tfrac{1}{3})-\frac{\alpha^3}{20}+\frac{\alpha^5}{420}\Gamma(\tfrac{2}{3})-\frac{\alpha^7}{8100}\Gamma(\tfrac{1}{3})+\dots\right]. \tag{14}$$

In the integral $\frac{1}{\pi}\int_0^\infty e^{-n(\sinh\theta+\theta)}\,d\theta$, on substituting t for $\sinh\theta+\theta$, reversing the series and proceeding as before, we get

$$\frac{1}{\pi}\int_0^\infty e^{-n(\sinh\theta+\theta)}\,d\theta=\frac{1}{\pi}\left\{\frac{1}{(2n)}-\frac{1}{2\,(2n)^3}+\frac{2}{(2n)^5}-\dots\right\}. \tag{15}$$

It has been shown* that

$$G_n(n)=\tfrac{1}{4}\left\{\alpha\,.\,\Gamma(\tfrac{1}{3})+\frac{\alpha^5}{420}\Gamma(\tfrac{2}{3})-\frac{\alpha^7}{8100}\Gamma(\tfrac{1}{3})-\dots\right\}.$$

Therefore

$$\Omega_n(n)=-\frac{2}{3\pi}G_n(n)+\frac{1}{\pi}\left\{\frac{1}{(2n)}-\frac{1}{2\,(2n)^3}+\frac{2}{(2n)^5}-\dots\right\}$$
$$-\frac{1}{3\pi}\left\{\frac{3}{5\,(2n)}-\frac{10449}{134750\,(2n)^3}+\dots\right\}$$
$$=-\frac{1}{6\pi}\left\{\alpha\,.\,\Gamma(\tfrac{1}{3})-\frac{2\alpha^3}{5}+\frac{\alpha^5}{420}\Gamma(\tfrac{2}{3})-\frac{\alpha^7}{8100}\Gamma(\tfrac{1}{3})+\frac{15973\,\alpha^9}{9702000}-\dots\right\}. \tag{16}$$

For $\Omega_{n-1}(n)$, the integrals in (10) are

$$\frac{1}{\pi}\int_0^\infty e^{-n(\sinh\theta-\theta)}\,e^{-\theta}\,.\,d\theta \text{ and } \frac{1}{\pi}\int_0^\infty e^{-n(\sinh\theta+\theta)}\,e^{\theta}\,d\theta.$$

The factor $e^{\pm\theta}$ is expanded in a series of ascending powers of λ, *e.g.* in the first integral $e^{-\theta}=1-\lambda+\frac{9\lambda^2}{20}-\frac{\lambda^3}{10}+\frac{\lambda^4}{280}-\frac{9\lambda^5}{2800}+\dots$ and we obtain the result

$$\Omega_{n-1}(n)-\Omega_n(n)\doteq\frac{1}{6\pi}\left\{\alpha^2\Gamma(\tfrac{2}{3})+\frac{\alpha^4}{30}\Gamma(\tfrac{1}{3})-\frac{37\alpha^6}{1050}+\frac{23\alpha^8}{113400}\Gamma(\tfrac{2}{3})\right.$$
$$\left.-\frac{947\alpha^{10}}{74844000}\Gamma(\tfrac{1}{3})+\frac{4\alpha^{12}}{15875}-\dots\right\}$$
$$=-\frac{2}{3\pi}\left\{G_{n-1}(n)-G_n(n)+\frac{111}{350n^2}-\frac{209}{2560n^4}+\dots\right\}, \tag{17}$$

* Debye, 'Math. Ann.,' vol. 67, p. 557:

$$G_n(n)=-\frac{\pi}{2}Y_n(n)=-\frac{i\pi}{4}\left\{H_n^{(\ )}(n)-H_n^{(1)}(n)\right\}.$$

'Phil. Mag.,' vol. 31, p. 526 (June, 1916).

the last term in each expansion being given approximately. When x is equal to n and n is an even number, the recurrence formula

$$\Omega_{n-1}(x)+\Omega_{n+1}(x) = \frac{2n}{x}\Omega_n(x)+\frac{4\sin^2 n\pi/2}{\pi x}$$

reduces to $$\Omega_{n-1}(n)-\Omega_{n+1}(n) = 2\{\Omega_{n-1}(n)-\Omega_n(n)\}. \tag{18}$$

Schläfli's polynomials $S_n(x) = \int_0^\infty e^{-x\sinh\theta}(e^{n\theta}-(-1)^n e^{-n\theta})\,d\theta$ may be employed to simplify the calculation, since the integral in (13) is equal to $S_n(n)+2\int_0^\infty e^{-n(\sinh\theta+\theta)}\,d\theta.$

Then

$$\Omega_n(n) = -\frac{2}{\pi}\left\{G_n(n)-\tfrac{1}{2}S_n(n)-\left(\frac{1}{2n}-\frac{1}{2(2n)^3}+\frac{2}{(2n)^5}-\ldots\right)\right\}, \tag{19}$$

$$\Omega_{n-1}(n) = -\frac{2}{\pi}\left\{G_{n-1}(n)-\tfrac{1}{2}S_{n-1}(n)+\left(\frac{1}{2n}+\frac{1}{(2n)^2}+\frac{1}{2(2n)^3}-\ldots\right)\right\}. \tag{20}$$

and

$$\Omega_{n-1}(n)-\Omega_n(n) = -\frac{2}{\pi}\left\{[G_{n-1}(n)-G_n(n)]-\tfrac{1}{2}[S_{n-1}(n)-S_n(n)] +\left(\frac{1}{n}+\frac{1}{(2n)^2}-\frac{1}{(2n)^4}+\frac{11}{2(2n)^6}-\ldots\right)\right\}. \tag{21}$$

If n is not very large, the functions $S_{n-1}(n)$ and $S_n(n)$ can be found without difficulty from

$$S_n(x) = \sum_{s=0}^{s\leqq[(n-1)/2]}\frac{(n-s-1)!}{s!}\left(\frac{2}{x}\right)^{n-2s}. \tag{22}$$

Table of $\Omega_{2m-1}(2m)-\Omega_{2m+1}(2m) = \Delta$.

m.	Δ.	m.	Δ.	m.	Δ.
1	0·308478	6	0·093342	11	0·061822
2	0·196336	7	0·084049	12	0·058277
3	0·149466	8	0·076754	13	0·055198
4	0·122976	9	0·070849	14	0·052493
5	0·105667	10	0·065955	15	0·050095

The corresponding values of R for $m = 1$ to $m = 6$ are

m.	R.	m.	R.
1	0·4846	4	3·0907
2	1·2336	5	4·1495
3	2·1130	6	5·2784

Thus as the value of m increases, *i.e.* for the higher modes of free vibration, λ_m, the equivalent wave-length, approaches the value $2\pi a/m$.

In the general case, where the order and argument are nearly equal, when $x = n+\kappa$, κ being a small number, the integral

$$\int_0^\infty e^{-x\sinh\theta+n\theta}\,d\theta = \int_0^\infty e^{-x(\sinh\theta-\theta)}\,e^{-\kappa\theta}\,d\theta.$$

If $\sinh\theta-\theta = t$ and λ is written for $(6t)^{1/3}$, then $\theta = \lambda-\frac{\lambda^3}{60}+\frac{\lambda^5}{1400}-\ldots$ and $d\theta = \left(1-\frac{\lambda^2}{20}+\frac{\lambda^4}{280}-\ldots\right)d\lambda$.

Expressing $e^{-\kappa\theta}$ in terms of λ we get, after a little reduction,

$$\int_0^\infty e^{-x\sinh\theta+n\theta}\,d\theta = \frac{1}{3}\int_0^\infty e^{-xt}\,6^{1/3}t^{-2/3}\{\sigma_0-\sigma_1(6t)^{1/3}+\sigma_2(6t)^{2/3}$$
$$-\sigma_3(6t)+\sigma_4(6t)^{4/3}-\ldots\}dt$$
$$=\frac{1}{3}\left\{\sigma_0\left(\frac{6}{x}\right)^{1/3}\Gamma\left(\frac{1}{3}\right)-\sigma_1\left(\frac{6}{x}\right)^{2/3}\Gamma\left(\frac{2}{3}\right)+\sigma_2\left(\frac{6}{x}\right)\right.$$
$$\left.-\frac{\sigma^3}{3}\left(\frac{6}{x}\right)^{4/3}\Gamma\left(\frac{1}{3}\right)+\frac{2\sigma_4}{3}\left(\frac{6}{x}\right)^{5/3}\Gamma\left(\frac{2}{3}\right)-2\sigma_5\left(\frac{6}{x}\right)^2+\ldots\right\}.$$

The first seven values of σ_n are as follows,

$$\sigma_0 = 1.$$

$$\sigma_1 = \kappa.$$

$$\sigma_2 = \frac{\kappa^2}{2}-\frac{1}{20}.$$

$$\sigma_3 = \frac{\kappa^3}{6}-\frac{\kappa}{15}.$$

$$\sigma_4 = \frac{\kappa^4}{24}-\frac{\kappa^2}{24}+\frac{1}{280}.$$

$$\sigma_5 = \frac{\kappa^5}{120}-\frac{\kappa^3}{60}+\frac{43\kappa}{8400}.$$

$$\sigma_6 = \frac{\kappa^6}{720}-\frac{7\kappa^4}{1440}+\frac{\kappa^2}{288}-\frac{1}{3600}.$$

For the integral $\int_0^\infty e^{-x\sinh\theta-n\theta}\,d\theta = \int_0^\infty e^{-x(\sinh\theta+\theta)}\,e^{\kappa\theta}\,d\theta$, put $t = \sinh\theta+\theta$ and $\lambda = t/2$.

Then $\theta = \lambda-\frac{\lambda^3}{12}+\frac{\lambda^5}{60}-\ldots$ and $d\theta = \left(1-\frac{\lambda^2}{4}+\frac{\lambda^4}{15}-\ldots\right)d\lambda$.

Since

$$e^{\kappa\theta} = 1+\kappa\lambda+\frac{\kappa^2}{2}\lambda^2+\left(\frac{\kappa^3}{6}-\frac{\kappa}{12}\right)\lambda^3+\left(\frac{\kappa^4}{24}-\frac{\kappa^2}{12}\right)\lambda^4+\ldots,$$

$$\int_0^\infty e^{-x\sinh\theta-n\theta}\,d\theta = \frac{1}{2}\int_0^\infty e^{-xt}\left\{s_0+s_1\frac{t}{2}+s_2\frac{t^2}{4}+s_3\frac{t^3}{8}+\ldots\right\}dt$$

$$= \frac{s_0}{2x}+\frac{s_1}{(2x)^2}+\frac{2s_2}{(2x)^3}+\frac{6s_3}{(2x)^4}+\frac{24s_4}{(2x)^5}+\ldots,$$

where

$$s_0 = 1.$$

$$s_1 = \kappa.$$

$$s_2 = \frac{\kappa^2}{2}-\frac{1}{4}.$$

$$s_3 = \frac{\kappa^3}{6}-\frac{\kappa}{3}.$$

$$s_4 = \frac{\kappa^4}{24}-\frac{5\kappa^2}{24}+\frac{1}{12}.$$

$$s_5 = \frac{\kappa^5}{120}-\frac{\kappa^3}{12}+\frac{29\kappa}{240}.$$

$$s_6 = \frac{\kappa^6}{720}-\frac{7\kappa^4}{288}+\frac{119\kappa^2}{1440}-\frac{43}{1440}.$$

Finally, knowing the development of $Y_n(x)$ or $-\frac{2}{\pi}G_n(x)$ in series of descending powers of x, we obtain the result

$$\Omega_n(x) = -\frac{1}{2\pi}\{\sigma_0\alpha\Gamma(\tfrac{1}{3})-\sigma_1\alpha^2\Gamma(\tfrac{2}{3})-\sigma_3\alpha^4\Gamma(\tfrac{4}{3})+\sigma_4\alpha^5\Gamma(\tfrac{5}{3})-\ldots\}$$

$$+\frac{1}{3\pi}\{\sigma_0\alpha\Gamma(\tfrac{1}{3})-\sigma_1\alpha^2\Gamma(\tfrac{2}{3})+\sigma_2\alpha^3-\sigma_3\alpha^4\Gamma(\tfrac{4}{3})+\sigma_4\alpha^5\Gamma(\tfrac{5}{3})-2\sigma_5\alpha^6-\ldots\}$$

$$+\frac{\cos n\pi}{\pi}\{s_0\beta+s_1\beta^2+2s_2\beta^3+6s_3\beta^4+24s_4\beta^5+120s_5\beta^6+\ldots\},$$

where $\alpha = \left(\frac{6}{x}\right)^{1/3}$ and $\beta = \frac{1}{2x}$.

The coefficients σ_n are the same as those occurring in the asymptotic expansion of $J_n(x)$, but σ_2, σ_5, etc., disappear in the final expression owing to the zero factor $\sin\frac{(n+1)\pi}{3}$. The arguments of the Γ functions associated with these coefficients are integers.